



ELSEVIER

Journal of Geometry and Physics 43 (2002) 345–350

JOURNAL OF  
GEOMETRY AND  
PHYSICS

www.elsevier.com/locate/jgp

# Global hyperbolicity and completeness

Yvonne Choquet-Bruhat<sup>a,\*</sup>, Spiros Cotsakis<sup>b</sup>

<sup>a</sup> University of Paris VI, L.P.T.L. Tour 22-12, 4 Place Jussieu, 75252 Paris, France

<sup>b</sup> GEODYSYC, Department of Mathematics, University of the Aegean, Karlovassi 83 200, Samos, Greece

Received 3 January 2002

---

## Abstract

We prove global hyperbolicity of spacetimes under generic regularity conditions on the metric. We then show that these spacetimes are timelike and null geodesically complete if the gradient of the lapse and the extrinsic curvature  $K$  are integrable. This last condition is required only for the tracefree part of  $K$  if the universe is expanding.

© 2002 Elsevier Science B.V. All rights reserved.

MSC: 58J45; 83C05

Subj. Class.: Differential geometry; General relativity

Keywords: Global hyperbolicity; Completeness; General relativity

---

## 1. Introduction

The global existence problem in general relativity does not reduce to a global existence theorem for a solution of the Einstein equations with some choice of time coordinate. The physical problem is the existence for an infinite proper time. But the proper time depends on the observer, i.e. on the timelike line along which it is observed. The notorious singularity theorems consider as a singularity the timelike or null geodesic incompleteness of a spacetime. They are established under various geometric hypothesis. On the other hand, existence theorems for a solution of generic Cauchy problems for the Einstein equations with an infinite proper time of existence for a family of observers have been obtained but their timelike or null geodesic completeness has not been proved, at least explicitly, in dimensions greater than  $1 + 1$  (cf. [1,2]).

We give in this paper generic conditions under which a spacetime is globally hyperbolic and sufficient conditions under which it is timelike and null geodesically complete. In the special case of an expanding universe our sufficient conditions for future completeness can

---

\* Corresponding author.

E-mail addresses: ycb@ccr.jussieu.fr (Y. Choquet-Bruhat), skot@aegean.gr (S. Cotsakis).

be weakened, it applies in particular to the spacetimes constructed in [2]. If one would find necessary conditions for completeness along the lines we used to obtain sufficient conditions, they would give a generic singularity theorem, based on analysis.

## 2. Global hyperbolicity

We consider a spacetime  $(\mathcal{V}, g)$  with  $\mathcal{V} = \mathcal{M} \times \mathcal{I}$ ,  $\mathcal{I} = (t_0, \infty)$ , where  $\mathcal{M}$  is a smooth manifold of dimension  $n$  and  ${}^{(n+1)}g$  a Lorentzian metric which in the usual  $n + 1$  splitting reads

$${}^{(n+1)}g \equiv -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j, \quad \theta^0 = dt, \quad \theta^i \equiv dx^i + \beta^i dt. \tag{2.1}$$

The spacetime is time-oriented by increasing  $t$ . We have chosen  $\mathcal{I} = (t_0, \infty)$  because we had in mind the case of an expanding universe with a singularity in the past, for instance at  $t = 0 < t_0$ . However, since  $t$  is just a coordinate, our study could apply as well to any interval  $\mathcal{I} \subset \mathbb{R}$ . To avoid irrelevant writing complications we suppose the metric to be as smooth as necessary and we make the following hypotheses which exclude pathologies in the metric and its representative:

- **Bounded lapse:** the lapse function  $N$  is bounded below and above by positive numbers  $N_m$  and  $N_M$

$$0 < N_m \leq N \leq N_M. \tag{2.2}$$

This hypothesis insures that the parameter  $t$  measures, up to a positive factor bounded above and below, the proper time along the normal to the slices  $\mathcal{M}_t (= \mathcal{M} \times \{t\})$ .

- **Slice completeness:** the time dependent metric  $g_t \equiv g_{ij} dx^i dx^j$  is a complete Riemannian metric on  $\mathcal{M}_t (= \mathcal{M} \times \{t\})$ , uniformly bounded below for all  $t \in \mathcal{I}$  by a metric  $\gamma = g_{t_0}$ . That is we suppose that there exists a number  $A > 0$  such that for all tangent vectors  $v$  to  $\mathcal{M}$  it holds that

$$A\gamma_{ij}v^i v^j \leq g_{ij}v^i v^j. \tag{2.3}$$

- **Uniformly bounded shift:** The  $g_t$  norm of the shift vector  $\beta$ , projection on the tangent space to  $\mathcal{M}_t$  of the tangent to the lines  $\{x\} \times \mathcal{I}$ , is uniformly bounded by a number  $B$ .

Under these hypotheses we prove the following theorem.

**Theorem 2.1.** *The spacetime  $(\mathcal{V} \equiv \mathcal{M} \times \mathcal{I}, {}^{(n+1)}g)$  is globally hyperbolic.*

We will use Geroch’s criterion of global hyperbolicity [3], equivalent to Leray’s original definition [4]. We will show that each  $\mathcal{M}_t$ ,  $t \in \mathcal{I}$ , is a Cauchy surface of the spacetime. In fact  $\mathcal{M}_t$  is a spacelike submanifold because its normal  $n$  is timelike: if  $n$  is future-directed its components in our frame are,  $n_0 = -N$ ,  $n_i = 0$ .

**Proof.** We now prove that  $\mathcal{M}_t$  is cut once by every inextendible causal curve. Indeed, if  $C : \mathcal{I} \rightarrow \mathcal{V} : \lambda \mapsto C(\lambda)$  is a future-directed causal curve, its tangent is such that,

$${}^{(n+1)}g \left( \frac{dC}{d\lambda}, n \right) \equiv -N \frac{dt}{d\lambda} < 0, \tag{2.4}$$

therefore on  $C$  we have that,

$$\frac{dt}{d\lambda} > 0, \tag{2.5}$$

and hence  $C$  can be reparametrized using  $t$  and cuts each  $\mathcal{M}_t$  at most once.

To show that a future-directed causal curve  $C$  issued from a point  $(x_1, t_1)$  cuts each  $\mathcal{M}_t$  with  $t > t_1$  we prove that any such curve can be extended to  $t = +\infty$  (an analogous reasoning will prove that a past-directed causal curve cuts each  $\mathcal{M}_t$  with  $t_0 < t < t_1$ ). Indeed, suppose the curve is defined for  $t \in [t_1, T)$ , with  $T$  finite (respectively  $T > t_0$ ). Consider a Cauchy sequence of numbers  $(t_n)$  which converges to  $T$  and the corresponding points  $(c_n, t_n)$  of the curve  $C$ , where  $c_n$  (with components  $C^i(t_n)$ ) are points of  $M$ . To show that, when  $(t_n)$  is a Cauchy sequence converging to  $T$ , these points converge to a limit point, denoted by  $c(T)$ , we consider the distance  $d$  in the complete metric space  $(\mathcal{M}, \gamma)$  and we have:

$$d(c_n, c_m) \leq \int_{t_n}^{t_m} \left( \gamma_{ij} \frac{dC^i}{dt} \frac{dC^j}{dt} \right)^{1/2} dt. \tag{2.6}$$

We deduce from hypothesis (2.3) that for all  $t$ ,

$$\gamma_{ij} \frac{dC^i}{dt} \frac{dC^j}{dt} \leq A^{-1} g_{ij} \frac{dC^i}{dt} \frac{dC^j}{dt}, \tag{2.7}$$

and from the causality of  $C$  and assumption (2.2) we obtain,

$$\left( g_{ij} \left( \frac{dC^i}{dt} + \beta^i \right) \left( \frac{dC^j}{dt} + \beta^j \right) \right)^{1/2} \leq N \leq N_M. \tag{2.8}$$

The classical subadditivity of norms implies then that,

$$\left( g_{ij} \frac{dC^i}{dt} \frac{dC^j}{dt} \right)^{1/2} \leq N + (g_{ij} \beta^i \beta^j)^{1/2} \leq N_M + B. \tag{2.9}$$

Assembling these results we find that,

$$d(c_n, c_m) \leq A^{-1}(N_M + B)(t_m - t_n). \tag{2.10}$$

This inequality shows that if  $(t_n)$  is a Cauchy sequence of numbers converging to  $T$ , the sequence  $(c_n)$  is a Cauchy sequence in the complete Riemannian manifold  $(\mathcal{M}, \gamma)$ , hence admits a limit point which we call  $c(T)$ . The timelike curve on  $[t_1, T)$  is therefore extendible. □

### 3. Completeness

We have seen that all future-directed timelike curves issued at time  $t_1$  can be represented by a mapping  $C : [t_1, +\infty) \rightarrow V$ . The proper length of such a curve is

$$\ell(C) \equiv \int_{t_1}^{+\infty} \left( N^2 - g_{ij} \left( \frac{dC^i}{dt} + \beta^i \right) \left( \frac{dC^j}{dt} + \beta^j \right) \right)^{1/2} dt, \tag{3.1}$$

and is therefore infinite if the lengths of the tangent vectors  $dC/dt$  at each point of  $C$  are bounded away from zero by a positive constant. The tangent vector to a curve depends on its parametrization. The use of  $t$  as a parameter on  $C$  is linked to the  $n + 1$  splitting. This motivates the following definition.

**Definition 3.1.** We say that the curve  $C$  is uniformly timelike relatively to the  $n + 1$  splitting if there exists a number  $k > 0$  such that on  $C$ ,

$$N^2 - g_{ij} \left( \frac{dC^i}{dt} + \beta^i \right) \left( \frac{dC^j}{dt} + \beta^j \right) \geq k^2. \tag{3.2}$$

We have obviously the following theorem.

**Theorem 3.1.** *The curves which are uniformly timelike relatively to the  $n + 1$  splitting have an infinite proper length.*

Examples of such curves are the orthogonal trajectories of the space sections  $M_t$ . The question of timelike geodesic completeness is more delicate since in an arbitrary spacetime these curves do not necessarily possess the uniformity mentioned above.

The tangent vector  $u$  to a geodesic parametrized by arc length, or by the canonical parameter in the case of a null geodesic, with components  $dx^\alpha/ds$  in the natural frame, satisfies in an arbitrary frame the differential equations

$$u^\alpha \nabla_\alpha u^\beta \equiv u^\alpha \partial_\alpha u^\beta + \omega_{\alpha\gamma}^\beta u^\alpha u^\gamma = 0. \tag{3.3}$$

In the adapted frame the components of  $u$  become,

$$u^0 = \frac{dt}{ds}, \quad u^i = \frac{dx^i}{ds} + \beta^i \frac{dt}{ds}, \tag{3.4}$$

while the Pfaff derivatives are given by,

$$\partial_0 \equiv \partial_t - \beta^i \partial_i, \quad \partial_i \equiv \frac{\partial}{\partial x^i}. \tag{3.5}$$

It holds therefore that,

$$u^\alpha \partial_\alpha u^\beta \equiv \frac{dt}{ds} \left( \partial_t - \beta^i \partial_i \right) u^\beta + \left( \frac{dx^i}{ds} + \beta^i \frac{dt}{ds} \right) \partial_i u^\beta \equiv \frac{du^\beta}{ds}. \tag{3.6}$$

Since  $u^0 \equiv dt/ds$ , Eq. (3.3) with  $\beta = 0$  can be written in the form,

$$\frac{d}{dt} \left( \frac{dt}{ds} \right) + \frac{dt}{ds} \left( \omega_{00}^0 + 2\omega_{0i}^0 v^i + \omega_{ij}^0 v^i v^j \right) = 0, \tag{3.7}$$

where we have set,

$$v^i = \frac{dx^i}{dt} + \beta^i. \tag{3.8}$$

If  $u$  is causal it holds that,

$$g_{ij} v^i v^j \leq N^2 \leq N_M^2, \tag{3.9}$$

and setting,

$$\frac{dt}{ds} = y, \tag{3.10}$$

Eq. (3.7) becomes,

$$\frac{y'}{y} = -(\omega_{00}^0 + 2\omega_{0i}^0 v^i + \omega_{ij}^0 v^i v^j). \tag{3.11}$$

The length (or canonical parameter extension) of the curve  $C$  is

$$\int_{t_1}^{+\infty} \frac{ds}{dt} dt, \tag{3.12}$$

and will be infinite if  $ds/dt$  is bounded away from zero i.e., if  $y \equiv dt/ds$  is uniformly bounded.

We deduce from (3.11) that,

$$\log \frac{y(t)}{y(t_1)} = - \int_{t_1}^t (\omega_{00}^0 + 2\omega_{0i}^0 v^i + \omega_{ij}^0 v^i v^j) dt. \tag{3.13}$$

The integrand on the right hand side is itself a function of  $t$  which depends on the integration of the geodesic equations. However, we can formulate *sufficient* conditions on the spacetime metric under which  $y$  is uniformly bounded using inequality (3.9). We denote by  $\nabla N$  the space gradient of the lapse  $N$ , by  $K_{ij} = -N\Gamma_{ij}^0$  the extrinsic curvature of  $M_t$  and by  $\tau \equiv g^{ij}K_{ij}$  the mean extrinsic curvature, negative in an expanding universe. We then have the following result on the future completeness of the spacetime  $(\mathcal{V}, g)$ .

**Theorem 3.2.** *Sufficient conditions for future timelike and null geodesic completeness of the metric (2.1) satisfying assumptions (2.2) and (2.3) are that, for each finite  $t_1$ :*

1.  $|\nabla N|_g$  is bounded by a function of  $t$  which is integrable on  $[t_1, +\infty)$ .
2.  $|K|_g$  is bounded by a function of  $t$  which is integrable on  $[t_1, +\infty)$ .

**Proof.** Using the expressions of the connection coefficients, Eq. (3.13) reads,

$$\log \frac{y(t)}{y(t_1)} = \int_{t_1}^t N^{-1}(-\partial_0 N - 2\partial_i N v^i + K_{ij} v^i v^j) dt. \tag{3.14}$$

On the other hand, we have on the curve  $C$ :

$$\partial_0 N = \frac{dN}{dt} - v^i \partial_i N, \tag{3.15}$$

which, using the inequality  $g_{ij}v^i v^j = |v|_g^2 \leq N^2$ , implies that,

$$\log \frac{y(t)}{y(t_1)} \leq 2 \log N_m^{-1} + N_m^{-1} \int_{t_1}^t (|\nabla N|_g N_M + |K|_g N_M^2) dt, \tag{3.16}$$

from which the result follows. □

**Corollary 3.3.** *Let  $P$  be the traceless part of  $K$ . Then in an expanding universe, Condition 2 can be replaced by*

- *Condition 2a: the norm  $|P|_g$  is integrable on  $[t_1, \infty)$ .*

**Proof.** If we set

$$K_{ij} = P_{ij} + \frac{1}{n} \tau g_{ij}, \quad (3.17)$$

we find that if the mean extrinsic curvature is negative,  $\tau \leq 0$ ,

$$K_{ij} v^i v^j \equiv P_{ij} v^i v^j + \frac{1}{n} \tau g_{ij} v^i v^j \leq P_{ij} v^i v^j, \quad (3.18)$$

and this completes the proof.  $\square$

If we could find *necessary* conditions for the integral (3.12) to be bounded, then we will have proved a generic singularity theorem, that is causal geodesic incompleteness.

#### 4. Example

The Lorentzian metric found in [2] on the three-manifold quotient of the spacetime by an  $S^1$  spacelike isometry group is of the type defined in Section 2 with  $\tau < 0$ ,

$$g_{ij} = e^{2\lambda} \sigma_{ij}, \quad (4.1)$$

where  $\sigma_t$  is uniformly equivalent to  $\sigma_{t_0}$ , for  $t_0 \geq 0$ , and

$$e^{2\lambda} \geq 2t^2, \quad 0 < N_m \leq N \leq 2. \quad (4.2)$$

The inequalities obtained in the quoted article show that Conditions 1 and 2a above are satisfied. The considered manifold is therefore future timelike and null geodesically complete.

#### References

- [1] D. Christodoulou, S. Klainerman, The Global Nonlinear Stability of the Minkowski Space, Princeton University Press, Princeton, NJ, 1993.
- [2] Y. Choquet-Bruhat, V. Moncrief, Future global in time Einsteinian spacetimes with  $U(1)$  isometry group, Ann. Henri Poincaré 2 (2001) 1001–1064.
- [3] R. Geroch, Domain of dependence, J. Math. Phys. 11 (1970) 437–449.
- [4] J. Leray, Hyperbolic Differential Equations, IAS, Princeton, 1952.